Resonance Analysis in 1D

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Dec 21, 1999

1 Phase Portraits

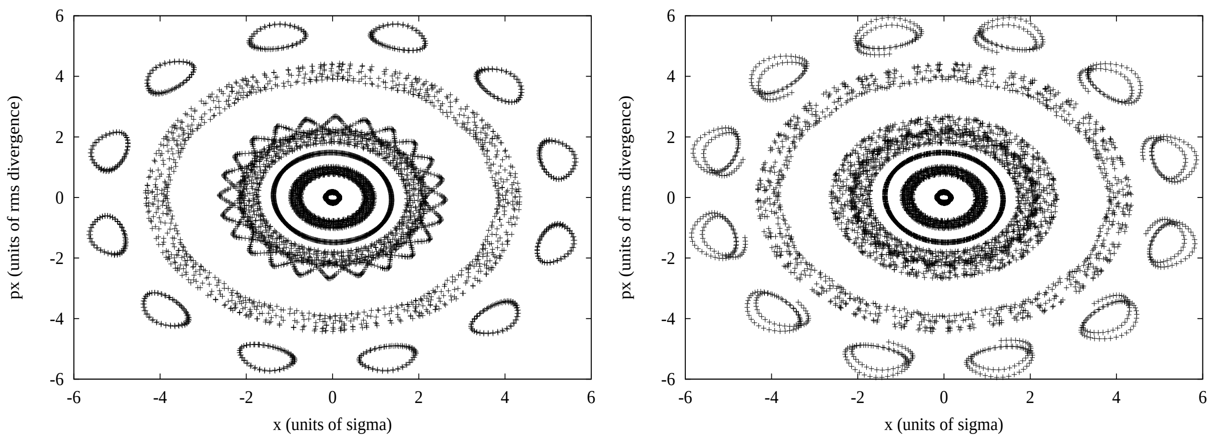

Figure 1: Horizontal phase space: **head-on interactions only**, 500μrad total crossing angle $Q_x = 0.581$, 7/12=0.583. Left figure: no time dependencies. Right figure: Tune modulation with amplitude=1×10^{-4}, frequency=35Hz.

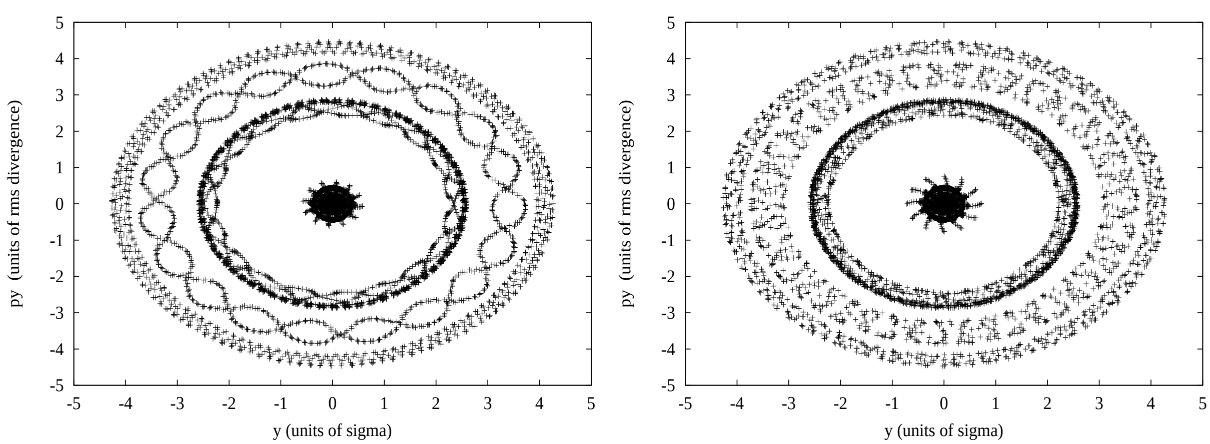

Figure 2: Vertical phase space: **head-on interactions only**, 500μrad total crossing angle $Q_y = 0.575$, 11/19=0.579. Left figure: no time dependencies. Right figure: Tune modulation with amplitude=1×10^{-4}, frequency=35Hz.

2 Analysis in 1 Degree of freedom

The 1D beam-beam potential is

$$U(x) = \frac{N_b r_{\bar{p}}}{\gamma_{\bar{p}}} \int_0^\infty \frac{1}{2\sigma_{op}^2 + q} \left[1 - e^{-x^2/(2\sigma_{op}^2+q)}\right] dq \tag{1}$$

Substituting

$$x = \sqrt{2\beta_x^* J_x} \cos\phi_x, \quad C = \frac{N_b r_{\bar{p}}}{\gamma_{\bar{p}}}$$

shows that the potential has the Fourier expansion

$$U = C\left[U_0 + \sum_{k=1}^{\infty} U_k \cos 2k\phi_x\right] \tag{2}$$

$$U_0(\frac{\beta_x^* J_x}{2\sigma_{op}^2}) = \int_0^{\beta_x^* J_x/(2\sigma_{op}^2)} \frac{dw}{w}[1 - e^{-w} I_0(w)] \tag{3}$$

is the detuning term and

$$U_k(\frac{\beta_x^* J_x}{2\sigma_{op}^2}) = (-1)^{k+1} 2 \int_0^{\beta_x^* J_x/(2\sigma_{op}^2)} \frac{dw}{w} e^{-w} I_k(w) \tag{4}$$

is a measure of the term driving the $2k$th order resonance. The complete Hamiltonian is

$$H(J_x, \phi_x) = \nu_x J_x + U(J_x, \phi_x)\delta_P(\theta) \tag{5}$$

$\delta_P(\theta)$ is the periodic delta function with period $2\pi/N_{IP}$, N_{IP} being the number of `uniformly distributed` intersection points.

$$\delta_P(\theta) = \frac{N_{IP}}{2\pi} \sum_{p=-\infty}^{\infty} e^{-ipN_{IP}\theta}$$

The equations of motion are

$$\frac{d\phi_x}{d\theta} = \nu_x - \frac{CN_{IP}}{2\pi}[U_0'(J_x) + \frac{1}{2}\sum_{k=-\infty}^{\infty} U_k'(J_x) e^{2ik\phi_x}] \sum_{p=-\infty}^{\infty} e^{-ipN_{IP}\theta} \tag{6}$$

$$\frac{dJ_x}{d\theta} = -\frac{CN_{IP}}{2\pi} i \sum_{k=-\infty}^{\infty} kU_k(J_x) e^{2ik\phi_x} \sum_{p=-\infty}^{\infty} e^{-ipN_{IP}\theta} \tag{7}$$

2.1 The Slowly Varying Hamiltonian

Assuming that the tune is close to one resonance we will average over the fast oscillating terms in the above Hamiltonian. and retain only the corresponding phase in the averaged Hamiltonian. The averaged Hamiltonian is

$$\bar{H} = \nu_x J_x + \frac{CN_{IP}}{2\pi}[U_0(J_x) + U_k(J_x)\cos(2k\phi_x - pN_{IP}\theta)] \tag{8}$$

Transforming to a frame rotating with the resonant tune $pN_{IP}/2k$ yields the new Hamiltonian

$$H = \delta J + \frac{CN_{IP}}{2\pi}[U_0(J) + U_k(J)\cos(2k\psi] \tag{9}$$

where δ is the distance (in tune space) to the resonance,

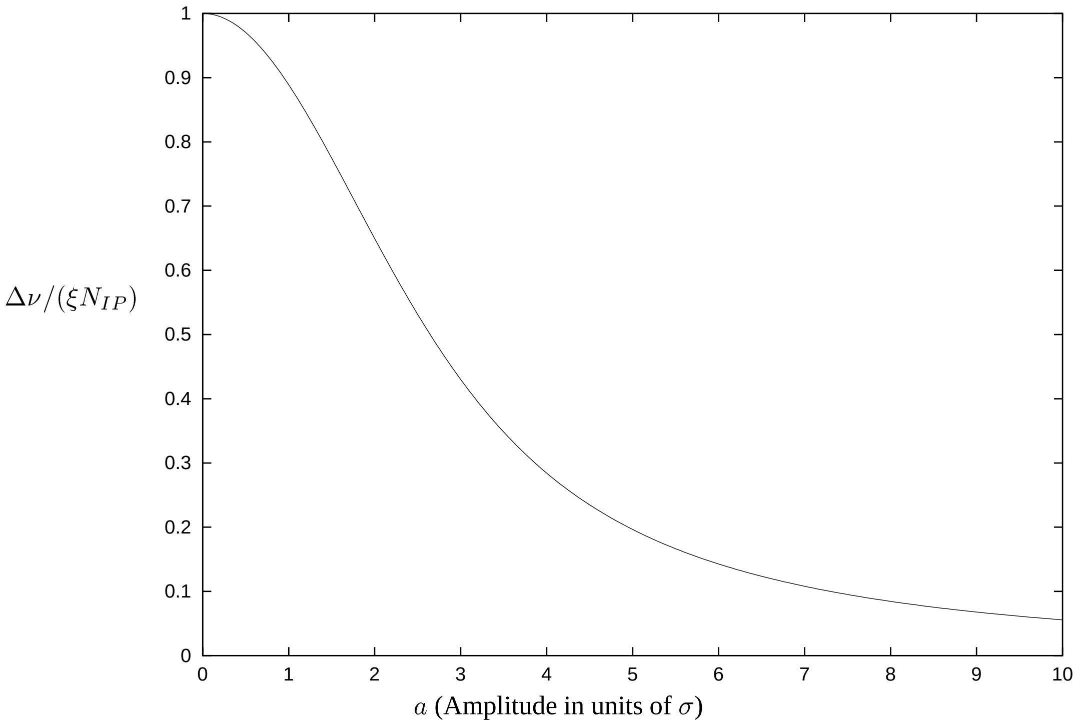

Figure 3: Scaled tune shift as a function of amplitude a

2.1.1 Tune shift and resonance driving terms

Define

$$A = \frac{N_b r_{\bar{p}} N_{IP}}{2\pi\gamma_{\bar{p}}}, \quad a = \frac{\sqrt{2\beta_x^* J}}{\sigma_{\bar{p}}}, \quad x = a\sigma_{\bar{p}}\cos\phi_x.$$

The tune shift parameter for anti-protons assuming round beams is

$$\xi = \frac{N_b r_{\bar{p}} \beta^*}{4\pi\gamma_{\bar{p}}\sigma_{op}^2} = \frac{N_b r_{\bar{p}}}{4\pi\gamma_{\bar{p}}\epsilon_{\bar{p}}} \tag{10}$$

The tune shift with amplitude in the lab frame is

$$\Delta\nu = \xi N_{IP} \frac{1 - e^{-(a\sigma_{\bar{p}}/(2\sigma_{op}))^2} I_0((a\sigma_{\bar{p}}/(2\sigma_{op}))^2)}{(a\sigma_{\bar{p}}/(2\sigma_{op}))^2} \tag{11}$$

$$\lim_{a\to 0}\Delta\nu = \xi N_{IP}, \qquad \lim_{a\to\infty}\Delta\nu = 0 \tag{12}$$

At large a, the detuning amplitude decreases as $\Delta\nu \propto 1/a^2$.

The resonance driving amplitude A_{2k} is

$$A_{2k} = AU_k'(J) = 8\xi N_{IP}\frac{\sigma_{op}^2}{\sigma_{\bar{p}}^2}\frac{e^{-(a\sigma_{\bar{p}}/(2\sigma_{op}))^2} I_k((a\sigma_{\bar{p}}/(2\sigma_{op}))^2)}{a^2} \tag{13}$$

It vanishes both at the origin and at infinity,

$$\lim_{a\to 0} A_{2k} = 0, \qquad \lim_{a\to\infty} A_{2k} = 0 \tag{14}$$

At large a, $A_{2k} \propto 1/a^3$, i.e. it decreases faster than the detuning amplitude.

2.1.2 Resonance Action and Fixed Points

The total tune (linear tune + detuning term) at the resonance amplitude equals the resonance tune $pN_{IP}/(2k)$.

$$\text{Resonant Action :} \qquad \delta + \Delta\nu(J_R) = \delta + AU_0'(J_R) = 0 \tag{15}$$

or the resonant amplitude $a_R = \sqrt{2\beta^* J_R}/\sigma_{\bar{p}}$ for a $2k$th order resonance is (assuming $\sigma_{\bar{p}} = \sigma_{op}$)

$$\boxed{\delta + \frac{2\beta^* A}{\sigma_{\bar{p}}^2 a_R^2}[1 - e^{-(a_R^2/4)} I_0(\frac{a_R^2}{4})] = 0} \tag{16}$$

The equations for the fixed points (ψ_f, J_f) are

$$0 = \sin 2k\psi_f \quad \Rightarrow \psi_f = \frac{n\pi}{2k}, n = 0, 1, \dots 4k-1 \tag{17}$$

$$\text{Fixed points :} \qquad 0 = \delta + AU_0'(J_f) + AU_k'(J_f)\cos(2k\psi_f) \tag{18}$$

The stability of the fixed points are as

$\cos 2k\psi_f = -1$	Stable if k is even,	Unstable if k is odd.
$\cos 2k\psi_f = 1$	Unstable if k is even	Stable if k is odd.

Stable fixed points for any k are at larger actions than the resonant action J_R
Unstable fixed points are at smaller actions than J_R.

However the difference between the fixed point actions and the resonance action will be small.

2.1.3 Island Width and Island Tune

The half width of the resonance island in J is

$$\boxed{\Delta J_w = 2\sqrt{\frac{U_k(J_s)}{U_0''(J_s)}}} \tag{19}$$

Caveats:
i) This expression for ΔJ is valid as long as $\Delta J \ll J_s$.
ii) The resonances which have been neglected to obtain the 1 resonance Hamiltonian must be far away.

The island width measured in tune units (or half the stopband width)

$$\boxed{\Delta\nu_w \simeq AU_0''(J_s)\Delta J_w = 2A\sqrt{U_0''(J_s)U_k(J_s)}} \tag{20}$$

The tune of small amplitude oscillations in the island is

$$\boxed{\nu_{Is} = 2kA\sqrt{|U_0''(J_s)U_k(J_s)|}} \tag{21}$$

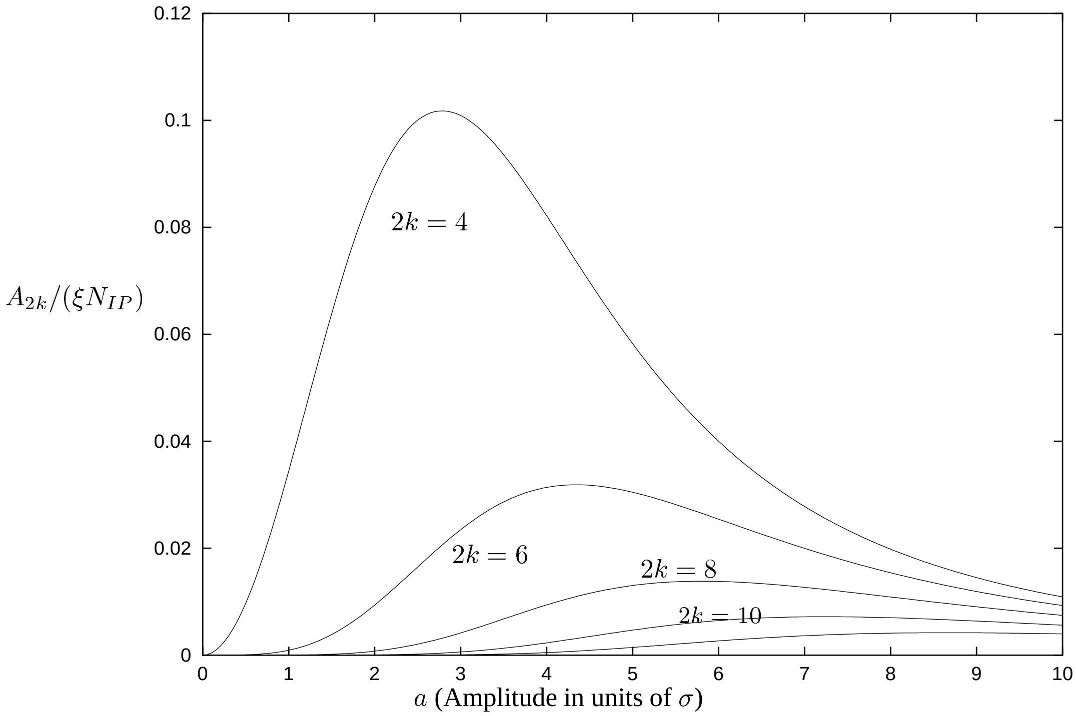

Figure 4: Resonance Order $2k$ driving amplitude as a function of amplitude a

Tevatron Parameters

Tunes Q_x, Q_y	0.581, 0.575
Beam-beam tune shift/IP for $\bar{p}$	9.89×10^{-3}
Number of interaction points N_{IP}	2

12th order resonances (horizontal phase space)

Resonant amplitude a_R	5.37 $\sigma_{\bar{p}}$
(Stable, unstable) fixed points	(5.46, 5.29) $\sigma_{\bar{p}}$
Resonance half width in amplitude	0.599 $\sigma_{\bar{p}}$
Resonance half width in tune	4.77×10^{-4}
Island frequency	**136 Hz**

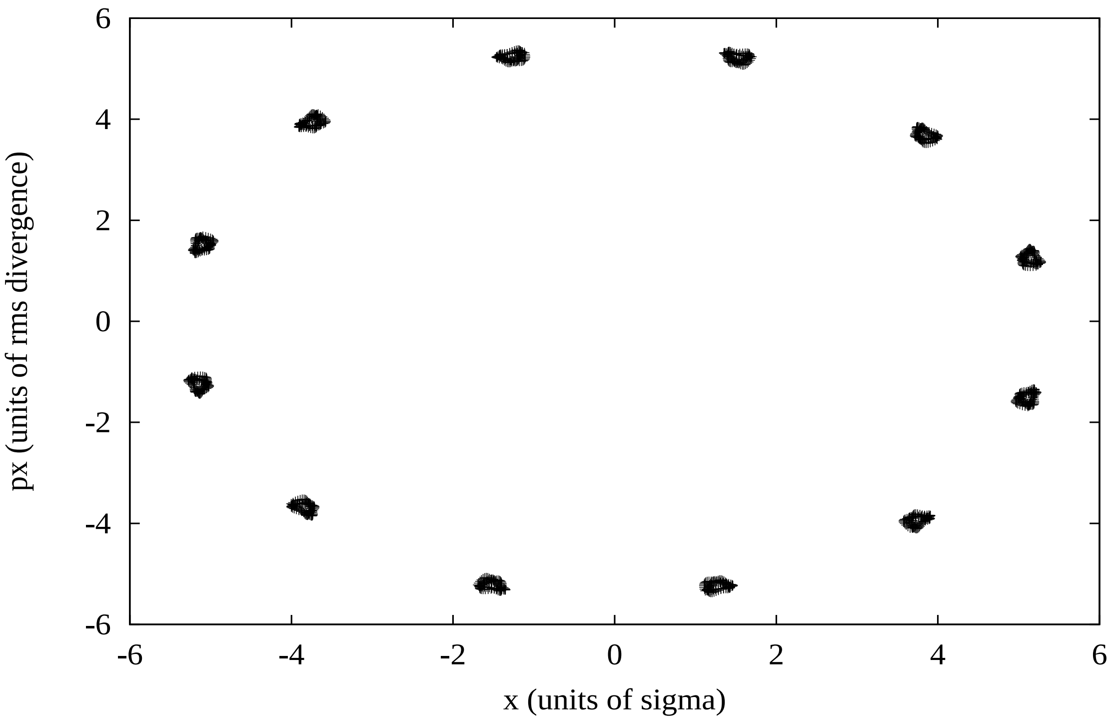

Figure 5: Horizontal phase portrait with initial conditions: $x_0 = 5.4\sigma_{x,\bar{p}} \cos \pi/12$,$x'_0 = -5.4\sigma_{x',\bar{p}} \sin \pi/12$, initial y amplitude a_y = 0.4. The islands are very close to the locations predicted by the 1D resonance analysis. The islands persist at this x amplitude until $a_y \leq 1.5$. At higher y amplitudes, the resonance amplitude in x decreases, e.g. at $a_y = 2.0$, the resonance amplitude in x is $a_R \approx 5.0$.